ANTI-COMMUTING REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract

In this paper we give a nonexistence theorem for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator.


Keywords and phrases: complex two-plane Grassmannians, hypersurfaces of type $B$, anti-commuting, contact hypersurface.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms there have been many characterizations of model hypersurfaces of type $A_1$, $A_2$, $B$, $C$, $D$ and $E$ in complex projective space $P_m(\mathbb{C})$, of type $A_0$, $A_1$, $A_2$ and $B$ in complex hyperbolic space $H_m(\mathbb{C})$, or of type $A_1$, $A_2$ and $B$ in quaternionic projective space $\mathbb{H}P^m$, which are completely classified by Cecil and Ryan [5], Kimura [6], Berndt [1], and Martinez and Pérez [7], respectively. Among them there have been only a few characterizations of homogeneous hypersurfaces of type $B$ in complex projective space $P_m(\mathbb{C})$. For example, the condition that $A\phi + \phi A = k\phi$, for nonzero constant $k$, is a model characterization of this kind of type $B$, which is a tube over a real projective space $\mathbb{R}P^n$ in $P_m(\mathbb{C})$, $m = 2n$ (see Yano and Kon [9]).

Let $M$ be a $(4m - 1)$-dimensional Riemannian manifold with an almost contact structure $(\phi, \xi, \eta)$ and an associated Riemannian metric $g$. Write

$$\omega(X, Y) = g(\phi X, Y), \quad (0.1)$$

where $\omega$ defines a 2-form on $M$ and rank $\omega = \text{rank } \phi = 4m - 2$. 

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If there is a nonzero-valued function \( \rho \) such that
\[
\rho g(\phi X, Y) = \rho \omega(X, Y) = d\eta(X, Y),
\]
the rank of the matrix \((\omega)\) being \(4m - 2\),
\[
\eta \wedge \omega \wedge \cdots \wedge \omega = \eta \wedge \rho^{-2m+1} \eta \wedge \cdots \wedge d\eta \neq 0.
\]

Let us denote by \(G_2(C^{m+2})\) the set of all complex two-dimensional linear subspaces of \(C^{m+2}\). We call such a set \(G_2(C^{m+2})\) complex two-plane Grassmannians. This Riemannian symmetric space \(G_2(C^{m+2})\) has a remarkable geometry that is equipped with both a Kähler structure \(J\) and a quaternionic Kähler structure \(\mathcal{J} = \text{Span}\{J_1, J_2, J_3\}\) not containing \(J\). In other words, \(G_2(C^{m+2})\) is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold (see Berndt and Suh [3, 4]).

Now we consider a \((4m - 1)\)-dimensional real hypersurface \(M\) in complex two-plane Grassmannians \(G_2(C^{m+2})\). Then from the Kähler structure of \(G_2(C^{m+2})\) there exists an almost contact structure \(\phi\) on \(M\). If the nonzero function \(\rho\) satisfies (0.2), we call \(M\) a contact hypersurface of the Kähler manifold. Moreover, it can easily be proved that a real hypersurface \(M\) in \(G_2(C^{m+2})\) is contact if and only if there exists a nonzero constant function \(\rho\) defined on \(M\) such that
\[
\phi A + A\phi = k\phi, \quad k = 2\rho.
\]
This means that
\[
g((\phi A + A\phi)X, Y) = 2d\eta(X, Y),
\]
where the exterior derivative \(d\eta\) of the 1-form \(\eta\) is defined by
\[
d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X
\]
for any vector fields \(X, Y\) on \(M\) in \(G_2(C^{m+2})\).

On the other hand, in \(G_2(C^{m+2})\) we are able to consider two kinds of natural geometric condition for real hypersurfaces \(M\) that
\[
[\xi] = \text{Span}\{\xi\} \quad \text{or} \quad \mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}, \quad \xi_i = -J_i N, \quad i = 1, 2, 3,
\]
where \(N\) denotes a unit normal to \(M\), is invariant under the shape operator \(A\) of \(M\) in \(G_2(C^{m+2})\). The first result in this direction is the classification of real hypersurfaces in \(G_2(C^{m+2})\) satisfying both conditions. Namely, Berndt and Suh [3] have proved the following.
**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if:

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$; 

or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

In Theorem A the vector $\xi$ contained in the one-dimensional distribution $[\xi]$ is said to be a Hopf vector when it becomes a principal vector for the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$. Moreover, in such a situation $M$ is said to be a Hopf hypersurface.

Besides this, a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ also admits the three-dimensional distribution $D^\perp$, which is spanned by almost contact three-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathcal{D} \oplus \mathcal{D}^\perp$. Also Berndt and Suh [4] have given a characterization of real hypersurfaces of type A when the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor $\phi$, which is equivalent to the condition that the Reeb flow on $M$ is isometric, as follows.

**Theorem B.** Let $M$ be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

On the other hand, as a characterization of real hypersurfaces of type B in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ in Theorem A, Suh [8], asserted the following fact in terms of the contact hypersurface.

**Theorem C.** Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature satisfying

$$A\phi + \phi A = k\phi,$$

where the function $k$ is nonzero and constant. Then $M$ is congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

Now in this paper let us consider a real hypersurface $M$ in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ satisfying $A\phi + \phi A = 0$. When the function $k$ mentioned in Theorem C identically vanishes, the shape operator is said to be anti-commuting, that is, the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$ satisfies

$$A\phi + \phi A = 0. \quad (\star)$$

In such a case we call a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ satisfying $(\star)$ an anti-commuting hypersurface. We give a nonexistence property of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator as follows.

**Theorem.** There exist no anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature.
1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$; for details we refer to [2–4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form $\mathfrak{B}$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $\mathfrak{o} = e\mathfrak{k}$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $\mathfrak{B}$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $\mathfrak{B}$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is 8. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{CP}^3$ with constant holomorphic sectional curvature 8, we shall assume that $m \geq 2$ from now on. Note that the isomorphism $\text{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{r}$ induces a Kähler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{h}$ on $G_2(\mathbb{C}^{m+2})$. If $J_1$ is any almost Hermitian structure in $\mathfrak{h}$, then $JJ_1 = J_1J$, and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and tr$(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{h}$ consists of three local almost Hermitian structures $J_v$ in $\mathfrak{h}$ such that $p_1 J_{v+1} = J_{v+2} = -J_{v+1} J_v$, where the index is taken modulo 3. Since $\mathfrak{h}$ is parallel with respect to the Riemannian connection $\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{h}$ three local 1-forms $q_1, q_2, q_3$ such that

$$\nabla_X J_v = q_{v+2}(X) J_{v+1} - q_{v+1}(X) J_{v+2}$$

(1.1)

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and $W$ be a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that $W$ is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $jW \subset W$ for all $J \in \mathfrak{h}_p$. And we say that $W$ is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{h}$ of $\mathfrak{h}_p$ such that $jW \subset W$ for all $J \in \mathfrak{h}$ and $jW \perp W$ for all $J \in \mathfrak{h} \perp \subset \mathfrak{h}_p$. Here, the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{h}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{h}$ for which any local oriented orthonormal frame field of $\mathfrak{h}$ is a canonical local basis of $\mathfrak{h}$. A quaternionic (or totally complex) submanifold of
$G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (or totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$
\tilde{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\
- g(JX, Z)JY - 2g(JX, Y)JZ \\
+ \sum_{\nu=1}^{3} \{ g(J\nu X, Y)J\nu X \\
- g(J\nu X, Z)J\nu Y - 2g(J\nu X, Y)J\nu Z \} \\
+ \sum_{\nu=1}^{3} \{ g(J\nu JY, Z)J\nu JX - g(J\nu JX, Z)J\nu JY \},
$$

where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas

In this section let us give some basic formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which will be used later.

The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then expression (1.2) for the curvature tensor $\tilde{R}$, the Gauss and the Codazzi equations are respectively given by

$$
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y \\
+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
+ \sum_{\nu=1}^{3} \{ g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z \} \\
+ \sum_{\nu=1}^{3} \{ g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y \} \\
- \sum_{\nu=1}^{3} \{ \eta(\nu Y)\eta_\nu (Z)\phi_\nu \phi X - \eta(\nu X)\eta_\nu (Z)\phi_\nu \phi Y \} \\
- \sum_{\nu=1}^{3} \{ \eta(\nu X)g(\phi_\nu \phi Y, Z) - \eta(\nu Y)g(\phi_\nu \phi X, Z) \}\xi_\nu \\
+ g(AY, Z)AX - g(AX, Z)AY
$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
+ \sum_{\nu=1}^{3} \{ \eta_\nu (X)\phi_\nu Y - \eta_\nu (Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \}$$
$$+ \sum_{\nu=1}^{3} \{ \eta_{\nu}(\phi X)\phi_{\nu}Y - \eta_{\nu}(\phi Y)\phi_{\nu}\phi X \}$$
$$+ \sum_{\nu=1}^{3} \{ \eta(X)\eta_{\nu}(\phi Y) - \eta(Y)\eta_{\nu}(\phi X) \} \xi_{\nu},$$

where $R$ denotes the curvature tensor of a real hypersurface $M$ in $G_2(C^{m+2})$.

The following identities can be proved straightforwardly and will be used frequently in subsequent calculations:

$$\phi_{\nu+1}\xi_{\nu} = -\xi_{\nu+2}, \quad \phi_{\nu}\xi_{\nu+1} = \xi_{\nu+2},$$
$$\phi\xi_{\nu} = \phi_{\nu}\xi, \quad \eta_{\nu}(\phi X) = \eta(\phi_{\nu}X), \quad \eta_{\nu+1}(\phi Y) = \eta_{\nu}(\phi_{\nu+1}X)\xi_{\nu},$$
$$\phi_{\nu+1}\phi_{\nu}X = -\phi_{\nu+2}X + \eta_{\nu+1}(\phi Y)\xi_{\nu+1}. \quad (2.1)$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_{\nu}X = \phi_{\nu}X + \eta_{\nu}(X)N$$

for any tangent vector $X$ of a real hypersurface $M$ in $G_2(C^{m+2})$, where $N$ denotes a unit normal vector of $M$ in $G_2(C^{m+2})$. Then from this and formulas (1.1) and (2.1),

$$\nabla_X \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \quad (2.2)$$
$$\nabla_X \xi_{\nu} = q_{\nu+1}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_{\nu} AX, \quad (2.3)$$
$$\nabla_X \phi_{\nu}Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_{\nu}(Y)AX - g(AX, Y)\xi_{\nu}. \quad (2.4)$$

Summing up these formulas, we obtain

$$\nabla_X (\phi_{\nu}\xi) = \nabla_X (\phi\xi_{\nu})$$
$$= (\nabla_X \phi)\xi_{\nu} + \phi(\nabla_X \xi_{\nu})$$
$$= q_{\nu+2}(X)\phi_{\nu+1}\xi_{\nu} - q_{\nu+1}(X)\phi_{\nu+2}\xi_{\nu} + \phi_{\nu}\phi AX$$
$$- g(AX, \xi_{\nu})\xi_{\nu} + \eta(\xi_{\nu})AX. \quad (2.5)$$

Moreover, from $JJ_{\nu} = J_{\nu}J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_{\nu}X = \phi_{\nu}\phi X + \eta_{\nu}(X)\xi_{\nu} - \eta(\xi_{\nu})AX. \quad (2.6)$$

### 3. Some key propositions

Now let us take an inner product to Codazzi’s equation with $\xi$ and use (2.1) and (2.2). Then

$$g((\nabla_X A)Y, \xi) - g((\nabla_Y A)X, \xi) = -2g(\phi X, Y)$$
$$+ 2 \sum_{\nu=1}^{3} \{ \eta_{\nu}(X)\eta_{\nu}(\phi Y) - \eta_{\nu}(Y)\eta_{\nu}(\phi X) - g(\phi_{\nu}X, Y)\eta_{\nu}(\xi) \}. \quad (2.4)$$
On the other hand, from formula (*) in the introduction, $A\xi = \alpha \xi$ where $\alpha = \eta(A\xi)$. From this, by taking the covariant derivative and using (2.2),

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha \phi AX - A\phi AX.$$  

It follows that

$$g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) - 2g(A\phi AX, Y).$$

Combining the above two equations,

$$-2g(\phi X, Y) + 2 \sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\}$$

$$= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) - 2g(A\phi AX, Y). \tag{3.1}$$

Putting $X = \xi$ in (3.1),

$$Y\alpha = (\xi\alpha)\eta(Y) - 4 \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y), \tag{3.2}$$

$$\text{grad } \alpha = (\xi\alpha)\xi + 4 \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu. \tag{3.3}$$

Now substituting (3.2) into (3.1) gives

$$g(A\phi AX, Y) - g(\phi X, Y) = 2 \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi)$$

$$- \sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \tag{3.4}$$

for any tangent vector fields $X$ and $Y$ on $M$.

**Lemma 3.1.** Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator. Then $\text{Tr } A = \alpha$.

**Proof.** From (*) and (2.2) it follows that

$$AX - \phi A\phi X - \alpha \eta(X)\xi = 0,$$

where we have put $\alpha = \eta(A\xi)$. If we take an orthonormal basis for $M$ in such a way that

$$\{e_i \mid i = 1, 2, \ldots, 4m - 1\},$$

then

$$\sum_{i=1}^{4m-1}(g(Ae_i, e_i) - g(\phi A\phi e_i, e_i) - \alpha \eta(e_i)g(\xi, e_i)) = 0,$$

that is, $\text{Tr } A - \text{Tr } \phi A\phi - \alpha = 0$. 

On the other hand, we see that $\text{Tr} \phi A \phi = \text{Tr} A \phi^2 = -\text{Tr} A + \alpha$. Therefore, $\text{Tr} A = \alpha$. \hfill \Box

**Lemma 3.2.** Let $M$ be an anti-commuting real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature. Then $\xi$ belongs to either the distribution $\mathcal{D}$ or the distribution $\mathcal{D}^\perp$.

**Proof.** By Lemma 3.1 and the assumption we know that $\alpha$ is constant. And from (3.2) we get

$$\sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(\phi Y) = 0.$$ 

Now let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathcal{D}$ and $\xi_1 \in \mathcal{D}^\perp$. Then $$\eta_{1}(\xi) \eta_{1}(\phi Y) = 0.$$

First, if $\eta_{1}(\xi) = 0$, then obviously $\xi \in \mathcal{D}$.

Next let us consider the case where $\eta_{1}(\phi Y) = 0$. By putting $\phi_{1}\xi$ in $Y$ we know $\eta(X_0) = 0$, which gives $\xi \in \mathcal{D}^\perp$. This proves our assertion. \hfill \Box

Now let us denote by $\mathfrak{h}$ the orthogonal complement of the Reeb vector field $\xi$ in the tangent space of $M$ in $G_2(\mathbb{C}^{m+2})$.

**Lemma 3.3.** If $A \phi + \phi A = 0$, $X \in \mathfrak{h}$ with $AX = \lambda X$, then

$$\lambda A \phi X - \phi X + \sum_{\nu=1}^{3} \{ 2\eta_{\nu}(\xi) \eta_{\nu}(\phi X) \xi - \eta_{\nu}(X) \phi_{\nu} \xi - \eta_{\nu}(\phi X) \xi_{\nu} - \eta_{\nu}(\xi) \phi_{\nu} X \} = 0.$$ 

(3.5)

**Proof.** From (3.4) it follows that

$$A \phi AX - \phi X + 2 \sum_{\nu=1}^{3} \{ \eta(\xi) \phi_{\nu} \xi_{\nu} + \eta_{\nu}(\phi X) \xi \} \eta_{\nu}(\xi)$$

$$- \sum_{\nu=1}^{3} \{ \eta_{\nu}(X) \phi_{\nu} \xi_{\nu} + \eta_{\nu}(\phi X) \xi_{\nu} + \eta_{\nu}(\xi) \phi_{\nu} X \} = 0.$$ 

And using the assumption, for $X \in \mathfrak{h}$ such that $AX = \lambda X$, leads to the above formula. \hfill \Box

**Proposition 3.4.** There exist no anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature for $\xi \in \mathcal{D}^\perp$.

**Proof.** By (3.5) and (*), for any $X \in \mathfrak{h}$,

$$(\lambda^2 + 1)X - \sum_{\nu=1}^{3} \{ \eta_{\nu}(X) \phi_{\nu} \xi + \eta_{\nu}(\phi X) \phi_{\nu} \xi_{\nu} + \eta_{\nu}(\xi) \phi_{\nu} X \} = 0.$$
Since $\xi \in \mathcal{D}^\perp$, we can put $\xi = \xi_1$. Then
\[
(\lambda^2 + 1)X + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 - \phi\phi_1X = 0.
\]

Since $X \in \mathfrak{h}$, we suppose that $X = \mathcal{D}X + \eta_2(X)\xi_2 + \eta_3(X)\xi_3$. This implies that
\[
(\lambda^2 + 1)\mathcal{D}X + (\lambda^2 + 2)\eta_2(X)\xi_2 + (\lambda^2 + 2)\eta_3(X)\xi_3 - \phi\phi_1\mathcal{D}X = 0.
\tag{3.6}
\]

Putting $X = \xi_2$ and $X = \xi_3$ in (3.6), we obtain $(\lambda^2 + 2)\xi_2 = 0$ and $(\lambda^2 + 2)\xi_3 = 0$, respectively. From these facts, we see that $\lambda^2 + 2 = 0$. Therefore we get a contradiction, which gives the proof of our proposition. \hfill \Box

### 4. Anti-commuting hypersurfaces in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathcal{D}^\perp$

In this section we wish to show that there exist no hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator for $\xi \in \mathcal{D}$. In order to do this we assert the following result.

**Lemma 4.1.** Let $M$ be an anti-commuting real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature for $\xi \in \mathcal{D}$. Then $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$.

**Proof.** From the assumption we know that the function $\alpha$ is constant. Then for $\xi \in \mathcal{D}$ and from (3.1), for any tangent vector field $X$ on $M$,
\[
\phi X - A\phi AX + \sum_{v=1}^{3}\{\eta_v(X)\phi_v\xi + \eta_v(\phi X)\xi_v\} = 0.
\tag{4.1}
\]

To prove this lemma it suffices to show that $g(A\mathcal{D}, \xi_v) = 0$, $v = 1, 2, 3$. In order to do this, we put
\[
\mathcal{D} = [\xi] \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi] \oplus \mathcal{D}_0,
\]
where the distribution $\mathcal{D}_0$ is an orthogonal complement of $[\xi] \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi]$ in the distribution $\mathcal{D}$.

First, from the assumption $\xi \in \mathcal{D}$ we know that $g(A\xi, \xi_v) = 0$, $v = 1, 2, 3$, because $A\xi = \alpha\xi$.

Next, we also get the conclusion $g(A\phi_i\xi, \xi_v) = 0$, for $i, v = 1, 2, 3$. In fact, using (2.3) and $\xi \in \mathcal{D}$,
\[
g(A\phi_i\xi, \xi_v) = g(A\xi_v, \phi_i\xi)
= g(A\xi_v, \phi\xi_i)
= -g(\phi A\xi_v, \xi_i)
= -g(\nabla_{\xi_i} \xi, \xi_i)
= g(\xi, \nabla_{\xi_i} \xi_i)
= g(\xi, q_{i+2}(\xi_v)\xi_{i+1} - q_{i+1}(\xi_v)\xi_{i+2} + \phi_i A\xi_v)
= g(\xi, \phi_i A\xi_v)
= -g(A\phi_i\xi, \xi_v),
\]
that is, $g(A\phi_i\xi, \xi_v) = 0$, $v = 1, 2, 3$. 

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Finally, we consider the case \( X \in \mathcal{D}_0 \), where the distribution \( \mathcal{D}_0 \) is denoted by

\[ \mathcal{D}_0 = \{ X \in \mathcal{D} \mid X \perp \xi \text{ and } \phi_i \xi, \ i = 1, 2, 3 \}. \]

In order to show this, let us replace \( X \) by \( \xi \mu \) in (4.1). Then it follows that

\[ 2\phi \xi \mu = A\phi A\xi \mu. \]

From this, together with the assumption (*),

\[ A^2 \phi \xi \mu = -2\phi \xi \mu. \]

Then multiplying both sides by \( \phi \) and also using the formula \( A\phi + \phi A = 0 \),

\[ A^2(-\xi \mu + \eta(\xi \mu)\xi) = -2(-\xi \mu + \eta(\xi \mu)\xi). \]

This implies that

\[ A^2 \xi \mu = -2\xi \mu, \quad \mu = 1, 2, 3. \]  

(4.2)

On the other hand, if we consider the case where \( X \in \mathcal{D}_0 \) in (3.4), then

\[ \phi X = A\phi AX. \]

From \( A\phi + \phi A = 0 \), this becomes \(-A^2 \phi X = \phi X\). Then from this, replacing \( X \) by \( \phi X \) leads, for any \( X \in \mathcal{D}_0 \), to

\[ A^2 X = -X. \]  

(4.3)

Using (4.2) and (4.3),

\[ g(AX, \xi \mu) = g(A(-A^2 X), \xi \mu) = -g(A^3 X, \xi \mu) = -g(AX, A^2 \xi \mu) = -g(AX, -2\xi \mu) = 2g(AX, \xi \mu), \]

for any vector fields \( X \) in \( \mathcal{D}_0 \). Then for any \( X \in \mathcal{D}_0 \), \( g(AX, \xi \mu) = 0, \mu = 1, 2, 3 \). This completes the proof.

For a tube of type B in Theorem A let us recall a proposition given in Berndt and Suh [3] as follows.

**Proposition A.** Let \( M \) be a connected real hypersurface of \( G_2(\mathbb{C}^{m+2}) \). Suppose that \( A\mathcal{D} \subset \mathcal{D}, A\xi = \alpha \xi \), and \( \xi \) is tangent to \( \mathcal{D} \). Then the quaternionic dimension \( m \) of \( G_2(\mathbb{C}^{m+2}) \) is even, say \( m = 2n \), and \( M \) has five distinct constant principal curvatures

\[ \alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r), \]
with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathbb{J}J\xi, \quad T_\gamma = \mathbb{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathbb{J}T_\lambda = T_\lambda, \quad \mathbb{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Now by using Proposition A let us check whether a tube of type B in Theorem A, that is, a tube over a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$ cannot satisfy the formula ($\ast$).

In fact, for any $\xi, \nu \in T_\beta, \beta = 2 \cot 2r$, the eigenspace $T_\gamma = \mathbb{J}\xi$ gives $\phi \xi, \nu \in T_\gamma$. This implies that $A \phi \xi, 0$ for any $\nu = 1, 2, 3$. From this,

$$A \phi \xi, 0 + \phi A \xi, 0 = 2 \cot 2r \phi \xi, 0 = 0.$$

For any $X \in T_\lambda, \lambda = \cot r$, we know that $JT_\lambda = T_\mu$ gives

$$A \phi X + \phi AX = -\tan r \phi X + \cot r \phi X = 2 \cot 2r \phi X = 0.$$

From this, we get $\cot 2r = 0$, giving a contradiction. So real hypersurfaces of type B cannot satisfy formula ($\ast$).

**Proposition 4.2.** There exist no anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature for $\xi \in \mathfrak{D}$.

Taking this Proposition 4.2 together with Proposition 3.4 gives a complete proof of our main theorem in the introduction.

**References**


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